State-Space Model Inspired Multiple-Input Multiple-Output Spiking Neurons

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Background

State-Space Models (SSMs)

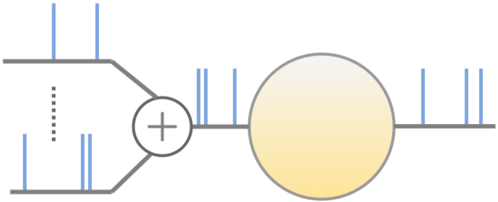
- long sequence modeling
- parallelization
- general dynamics
- continuous-valued

Spiking Neurons (SNNs)

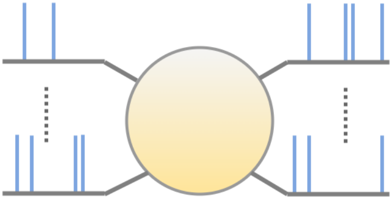
- sequence modeling
- recurrent non-linearity
- neuro-inspired dynamics
- spike-based

Our work – general linear SSM-based Multi-input Multi-output (MIMO) spiking neurons

Singe-Input Single-Output (SISO) vs Multi-Input Multi-Output (MIMO)



(a) Single-Input Single-Output (SISO) spiking neuron



(b) Multi-Input Multi-Output (MIMO) spiking neuron

Background - LIF model

The discrete-time LIF neuron:

$$u[t+1] = \alpha u[t] - \alpha\theta s_{out}[t] + (1-\alpha) i[t]$$

$$s_{out}[t] = f_{\theta}(u[t]) = \begin{cases} 1 \text{ if } u[t] \geq \theta \\ 0 \text{ otherwise.} \end{cases}$$

LIF model vs General

The discrete-time LIF neuron:

$$u[t+1] = \alpha u[t] - \alpha\theta s_{out}[t] + (1-\alpha)i[t]$$

$$s_{out}[t] = f_\theta(u[t]) = \begin{cases} 1 \text{ if } u[t] \geq \theta \\ 0 \text{ otherwise.} \end{cases}$$

A discrete-time general n-dimensional spiking neuron:

$$\boldsymbol{v}[t+1] = \boldsymbol{A}\boldsymbol{v}[t] - \boldsymbol{R}s_{out}[t] + \boldsymbol{B}i[t]$$

$$s_{out}[t] = f_{\boldsymbol{\Theta}}(\boldsymbol{v}[t]) = \begin{cases} 1 \text{ if } \boldsymbol{v}[t] \in \boldsymbol{\Theta} \\ 0 \text{ otherwise} \end{cases}$$

Parameter	LIF neuron	General neuron
State	u	$\boldsymbol{v}$
State transition	α	$\boldsymbol{A}$
Reset	$\alpha\theta$	$\boldsymbol{R}$
Input transition	$(1-\alpha)$	$\boldsymbol{B}$
Threshold	θ	$\boldsymbol{\Theta}$

Background - AdLIF model

Discrete-time adLIF is defined as follows

$$\begin{aligned} u[t+1] &= \alpha u[t] - \alpha\theta s_{out}[t] + (1-\alpha) i[t] - (1-\alpha) v[t] \\ v[t+1] &= a u[t] + \beta v[t] + b s_{out}[t] \\ s_{out}[t] &= f_{\theta}(u[t]) \end{aligned}$$

Re-written in matrix form:

$$\begin{bmatrix} u[t+1] \\ v[t+1] \end{bmatrix} = \begin{bmatrix} \alpha & -(1-\alpha) \\ a & \beta \end{bmatrix} \begin{bmatrix} u[t] \\ v[t] \end{bmatrix} + \begin{bmatrix} -\alpha\theta \\ b \end{bmatrix} s_{out}[t] + \begin{bmatrix} 1-\alpha \\ 0 \end{bmatrix} i[t]$$

AdLIF model vs General model

The discrete-time adLIF neuron:

$$\begin{bmatrix} u[t+1] \\ v[t+1] \end{bmatrix} = \begin{bmatrix} \alpha & -(1-\alpha) \\ a & \beta \end{bmatrix} \begin{bmatrix} u[t] \\ v[t] \end{bmatrix} + \begin{bmatrix} -\alpha\theta \\ b \end{bmatrix} s_{out}[t] + \begin{bmatrix} 1-\alpha \\ 0 \end{bmatrix} i[t]$$

$$s_{out}[t] = f_\theta(u[t])$$

A discrete-time general n-dim. neuron:

$$\boldsymbol{v}[t+1] = \boldsymbol{A}\boldsymbol{v}[t] - \boldsymbol{R}s_{out}[t] + \boldsymbol{B}i[t]$$

$$s_{out}[t] = f_{\boldsymbol{\Theta}}(\boldsymbol{v}[t]) = \begin{cases} 1 \text{ if } \boldsymbol{v}[t] \in \boldsymbol{\Theta} \\ 0 \text{ otherwise} \end{cases}$$

Parameter	LIF neuron	General neuron
State	$[u[t], v[t]]^T$	$\boldsymbol{v}$
State transition	$\begin{bmatrix} \alpha & -(1-\alpha) \\ a & \beta \end{bmatrix}$	$\boldsymbol{A}$
Reset	$\boldsymbol{R} = \begin{bmatrix} -\alpha\theta \\ b \end{bmatrix}$	$\boldsymbol{R}$
Input transition	$\begin{bmatrix} 1-\alpha \\ 0 \end{bmatrix}$	$\boldsymbol{B}$
Threshold	θ	$\boldsymbol{\Theta}$

General Spiking Model vs State Space Model (SSM)

A discrete-time general n-dim. neuron:

$$\boldsymbol{v}[t+1] = \boldsymbol{A}\boldsymbol{v}[t] - \boldsymbol{R}s_{out}[t] + \boldsymbol{B}i[t]$$

$$s_{out}[t] = f_{\boldsymbol{\Theta}}(\boldsymbol{v}[t]) = \begin{cases} 1 \text{ if } \boldsymbol{v}[t] \in \boldsymbol{\Theta} \\ 0 \text{ otherwise} \end{cases}$$

Parameter	Dimension
$\boldsymbol{A}$	$n \times n$
$\boldsymbol{B}$	$n \times 1$
$s_{out}[t]$	1×1

General Spiking Model vs State Space Model (SSM)

A discrete-time general n-dim. neuron:

$$\boldsymbol{v}[t+1] = \boldsymbol{A}\boldsymbol{v}[t] - \boldsymbol{R}s_{out}[t] + \boldsymbol{B}i[t]$$

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Parameter	Dimension
$\boldsymbol{A}$	$n \times n$
$\boldsymbol{B}$	$n \times 1$
$s_{out}[t]$	1×1

A discrete-time linear SSM can be written as

$$\boldsymbol{v}[t+1] = \boldsymbol{A}\boldsymbol{v}[t] + \boldsymbol{B}\boldsymbol{i}[t]$$
$$\boldsymbol{y}[t] = \boldsymbol{C}\boldsymbol{v}[t] + \boldsymbol{D}\boldsymbol{i}[t],$$

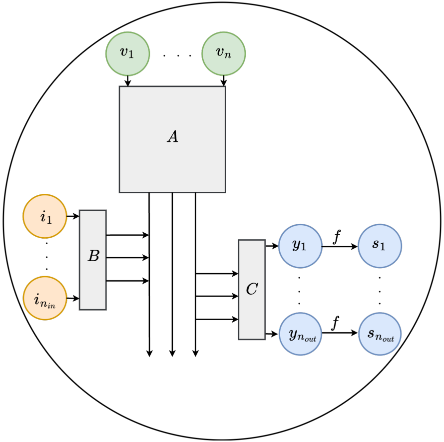
Parameter	Dimension
$\boldsymbol{A}$	$n \times n$
$\boldsymbol{B}$	$n \times n_{in}$
$\boldsymbol{C}$	$n_{out} \times n$
$\boldsymbol{D}$	$n_{out} \times n_{in}$
$\boldsymbol{y}[t]$	$n_{out} \times 1$

Proposed Model

Our Multi-input Multi-output (MIMO) spiking neuron is given by

$$\boldsymbol{v}[t+1] = \boldsymbol{A}\boldsymbol{v}[t] + \boldsymbol{B}\boldsymbol{i}[t]$$
$$\boldsymbol{y}[t] = \boldsymbol{C}\boldsymbol{v}[t] + \boldsymbol{c}_{bias}$$
$$\boldsymbol{s}_{out}[t] = f_\theta(\boldsymbol{y}[t])$$

Parameter	Dimension
$\boldsymbol{A}$	$n \times n$
$\boldsymbol{B}$	$n \times n_{in}$
$\boldsymbol{C}$	$n_{out} \times n$
$\boldsymbol{c}_{bias}$	$n_{out} \times 1$
$\boldsymbol{i}[t]$	$n_{in} \times 1$
$\boldsymbol{s}_{out}[t]$	$n_{out} \times 1$



Proposed Model vs General Spiking Model

Our Multi-input Multi-output (MIMO) neuron with n_{in} inputs and n_{out} outputs is given by

$$\begin{aligned} \boldsymbol{v}[t+1] &= \boldsymbol{A}\boldsymbol{v}[t] + \boldsymbol{B}\boldsymbol{i}[t] \\ \boldsymbol{y}[t] &= \boldsymbol{C}\boldsymbol{v}[t] + \boldsymbol{c}_{bias} \\ \boldsymbol{s}_{out}[t] &= f_{\theta}(\boldsymbol{y}[t]) \end{aligned}$$

Parameter	Dimension
$\boldsymbol{B}$	$n \times n_{in}$
$\boldsymbol{C}$	$n_{out} \times n$
$\boldsymbol{c}_{bias}$	$n_{out} \times 1$
$\boldsymbol{i}[t]$	$n_{in} \times 1$
$s_{out}[t]$	$n_{out} \times 1$

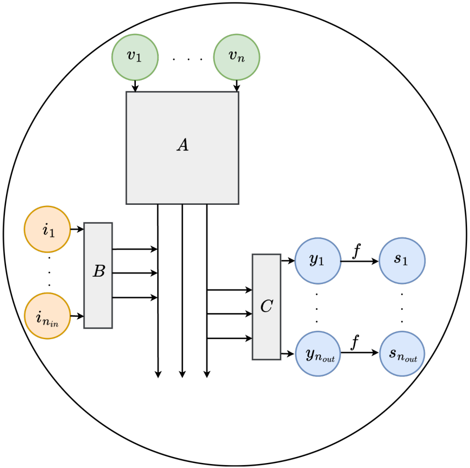
A discrete-time general n-dim. neuron:

$$\boldsymbol{v}[t+1] = \boldsymbol{A}\boldsymbol{v}[t] - \boldsymbol{R}s_{out}[t] + \boldsymbol{B}i[t]$$

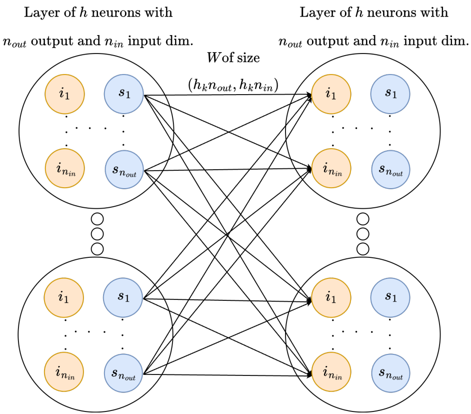
$$s_{out}[t] = f_{\Theta}(\boldsymbol{v}[t]) = \begin{cases} 1 \text{ if } \boldsymbol{v}[t] \in \boldsymbol{\Theta} \\ 0 \text{ otherwise} \end{cases}$$

Parameter	Dimension
$\boldsymbol{B}$	$n \times 1$
$i[t]$	1×1
$s_{out}[t]$	1×1

Illustration



(a) Single MIMO neuron.



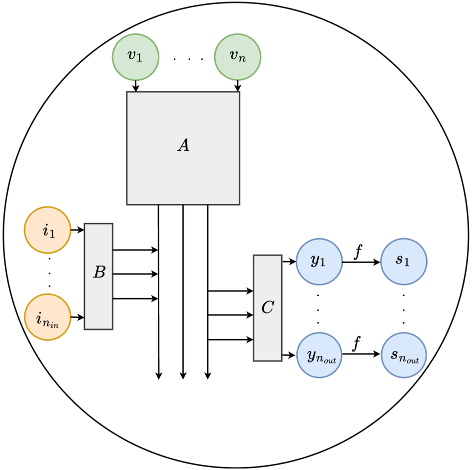
(b) An example network.

Research Questions

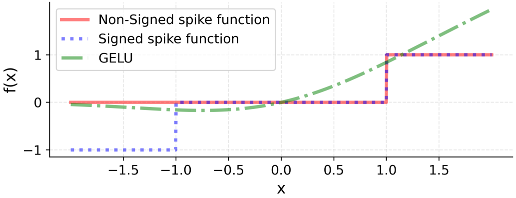
- Neuron state dimension vs number of neurons trade-offs?
- What structures of the state-transition matrices should be preferred?
- What should be the number of input and output channels?
- Performance lost in spiking-based neuron vs continuous-valued neuron?
- Can multi-output compensate for the information loss in spiking-based neuron?

Activation Function

When is the activation function f applied?



What activation functions are used in our work?



Experimental Set-Up

- SSM-Neuron Parametrization - Diagonal $\boldsymbol{A} = \boldsymbol{\Lambda}$ and Non-Diagonal $\boldsymbol{A} = \boldsymbol{Q}^H \boldsymbol{\Lambda} \boldsymbol{Q}$ where $\boldsymbol{Q}$ is the DFT matrix. In both $|\lambda_i| < 1$ for stability.

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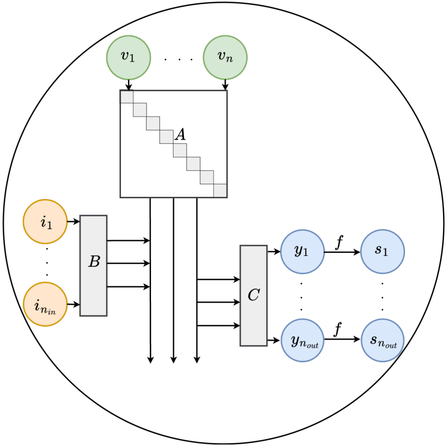
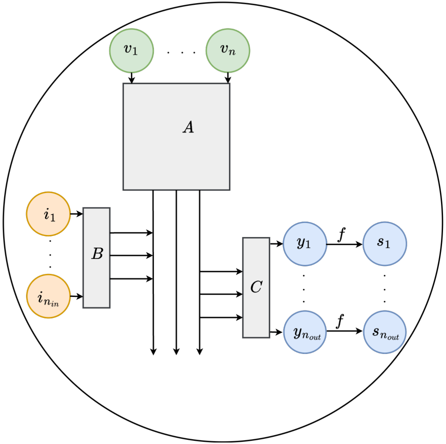


(a) Diagonal



(b) Non-Diagonal

Experimental Set-Up

- SSM-Neuron Parametrization - Diagonal $\boldsymbol{A} = \boldsymbol{\Lambda}$ and Non-Diagonal $\boldsymbol{A} = \boldsymbol{Q}^H \boldsymbol{\Lambda} \boldsymbol{Q}$ where $\boldsymbol{Q}$ is the DFT matrix. In both $|\lambda_i| < 1$ for stability.
- Types of models based on input/output dimensions

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- Types of models based on input/output dimensions

Parameter	Dimension
Single-Input Single-Output (SISO)	$n_{in} = 1, n_{out} = 1$
Multi-Input Multi-Output (MIMO)	$n_{in} > 1, n_{out} > 1$
Single-Input Multi-Output (SIMO)	$n_{in} = 1, n_{out} > 1$
Multi-Input Single-Output (MISO)	$n_{in} > 1, n_{out} = 1$

Experimental Set-Up

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- 2 hidden layer network trained using BPTT with surrogate gradient 30 epochs.

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- Types of models based on input/output dimensions
- 2 hidden layer network trained using BPTT with surrogate gradient 30 epochs.
- Dataset - Spiking Heidelberg Digits (SHD)

Number of Neurons vs Dimension of Neuron's State

Table: Number of hidden neurons h and state dimension of neurons n for $h \times n = 2048$, where 2 hidden layered model is used with SISO *signed*-spiking neurons and diagonal state transition matrix.

h	n	Accuracy
1024	2	88.1 ± 0.8 %
128	16	88.3 ± 1.3 %
64	32	87.4 ± 1.0 %
32	64	84.1 ± 1.6 %
16	128	74.1 ± 2.3 %
2	1024	12.0 ± 5.6 %

Effect of Increasing Input and/or Output Dimension

Table: Neurons with diagonal transition matrix and *signed*-spiking activation function used. Architecture used $h = 32$, $n = 64$.

Type	Input Dim	Output Dim	Accuracy
SISO	1	1	83.4 ± 1.2 %
SIMO	1	8	80.0 ± 0.8 %
	1	64	88.5 ± 1.6 %
	1	128	89.2 ± 1.4 %
MISO	8	1	38.3 ± 4.6 %
	64	1	34.3 ± 4.2 %
	128	1	36.8 ± 3.7 %
MIMO	8	8	62.3 ± 3.9 %
	64	64	72.9 ± 4.4 %
	128	128	75.6 ± 2.8 %

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Table Backgorund

Table: Architecture used $h = 32$, $n = 64$. SIMO with output dimension of 128.

State Transition Matrix	Single/Multi-Input/Output	Activation Function	Accuracy
Diagonal	SISO	Non-Sgn Spikes	79.9 ± 2.2 %
		Sgn Spikes	83.9 ± 0.9 %
		GELU	87.4 ± 1.1 %
Diagonal	SIMO	Non-Sgn Spikes	86.6 ± 1.2 %
		Sgn Spikes	89.5 ± 1.3 %
		GELU	90.4 ± 1.2 %
Non-Diagonal	SISO	Non-Sgn Spikes	79.9 ± 3.1 %
		Sgn Spikes	83.4 ± 1.6 %
		GELU	84.1 ± 1.9 %
Non-Diagonal	SIMO	Non-Sgn Spikes	83.6 ± 2.5 %
		Sgn Spikes	84.6 ± 1.3 %
		GELU	87.9 ± 1.3 %

Impact of Multiple-Output Channels

Table: Architecture used $h = 32$, $n = 64$. SIMO with output dimension of 128.

State Transition Matrix	Single/Multi-Input/Output	Activation Function	Accuracy
Diagonal	SISO	Non-Sgn Spikes	79.9 ± 2.2 %
		Sgn Spikes	83.9 ± 0.9 %
		GELU	87.4 ± 1.1 %
Diagonal	SIMO	Non-Sgn Spikes	86.6 ± 1.2 %
		Sgn Spikes	89.5 ± 1.3 %
		GELU	90.4 ± 1.2 %
Non-Diagonal	SISO	Non-Sgn Spikes	79.9 ± 3.1 %
		Sgn Spikes	83.4 ± 1.6 %
		GELU	84.1 ± 1.9 %
Non-Diagonal	SIMO	Non-Sgn Spikes	83.6 ± 2.5 %
		Sgn Spikes	84.6 ± 1.3 %
		GELU	87.9 ± 1.3 %

Table: Architecture used $h = 32$, $n = 64$. SIMO with output dimension of 128.

State Transition Matrix	Single/Multi-Input/Output	Activation Function	Accuracy
Diagonal	SISO	Non-Sgn Spikes	79.9 ± 2.2 %
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Diagonal	SIMO	Non-Sgn Spikes	86.6 ± 1.2 %
		Sgn Spikes	89.5 ± 1.3 %
		GELU	90.4 ± 1.2 %
Non-Diagonal	SISO	Non-Sgn Spikes	79.9 ± 3.1 %
		Sgn Spikes	83.4 ± 1.6 %
		GELU	84.1 ± 1.9 %
Non-Diagonal	SIMO	Non-Sgn Spikes	83.6 ± 2.5 %
		Sgn Spikes	84.6 ± 1.3 %
		GELU	87.9 ± 1.3 %

Coupling of State Variables of a Neuron

Table: Architecture used $h = 32$, $n = 64$. SIMO with output dimension of 128.

State Transition Matrix	Single/Multi-Input/Output	Activation Function	Accuracy
Diagonal	SISO	Non-Sgn Spikes	79.9 ± 2.2 %
		Sgn Spikes	83.9 ± 0.9 %
		GELU	87.4 ± 1.1 %
Diagonal	SIMO	Non-Sgn Spikes	86.6 ± 1.2 %
		Sgn Spikes	89.5 ± 1.3 %
		GELU	90.4 ± 1.2 %
Non-Diagonal	SISO	Non-Sgn Spikes	79.9 ± 3.1 %
		Sgn Spikes	83.4 ± 1.6 %
		GELU	84.1 ± 1.9 %
Non-Diagonal	SIMO	Non-Sgn Spikes	83.6 ± 2.5 %
		Sgn Spikes	84.6 ± 1.3 %
		GELU	87.9 ± 1.3 %

1. Background
2. Methods
3. Results
4. Conclusions

Conclusions

- Promising results can be achieved using our proposed general SSM-based MIMO neuron compared other neuro-inspired SNN results such as LIF and adLIF.
- Trade-offs among various parameters:
  - neuron state size vs number of neurons
  - number of input/output channels
  - activation functions
- In networks with low number of neurons with large internal states, using multiple-output channels may improve the performance significantly

Thank you!

Other tables

Additional table

Table: Input-Output dimension comparison and trade-off. Neurons with diagonal transition matrix and *signed*-spiking activation function used. Architecture used $h = 128$, $n = 16$.

Type	Input Dim	Output Dim	Accuracy
SISO	1	1	88.8 ± 1.0 %
SIMO	1	8	87.7 ± 1.0 %
	1	64	90.0 ± 0.7 %
	1	128	89.9 ± 0.8 %
MISO	8	1	59.8 ± 2.0 %
	64	1	52.3 ± 2.4 %
	128	1	54.2 ± 3.3 %
MIMO	8	8	75.5 ± 2.0 %
	64	64	76.5 ± 2.2 %
	128	128	78.0 ± 1.9 %

Additional table

Table: Architecture used $h = 128, n = 16$. SIMO with output dimension of 128.

State Transition Matrix	Single/Multi-Input/Output	Activation Function	Accuracy
Diagonal	SISO	Non-Sgn Spikes	87.4 ± 1.2 %
		Sgn Spikes	88.3 ± 0.7 %
		GELU	88.8 ± 0.7 %
Diagonal	SIMO	Non-Sgn Spikes	89.2 ± 1.0 %
		Sgn Spikes	90.2 ± 0.9 %
		GELU	89.5 ± 1.1 %
Non-Diagonal	SISO	Non-Sgn Spikes	86.1 ± 1.5 %
		Sgn Spikes	86.5 ± 1.0 %
		GELU	86.7 ± 1.5 %
Non-Diagonal	SIMO	Non-Sgn Spikes	86.1 ± 1.2 %
		Sgn Spikes	87.5 ± 1.8 %
		GELU	88.2 ± 0.7 %