Towards fractional order dynamics
neuromorphic elements

Fidel Santamaria, PhD
The University of Texas at San Antonio
NICE, La Jolla, CA April 24th, 2024

Fidel.Santamaria@utsa.edu
Challenge: A model that is used as theory

- The leaky integrate-and-fire

\[ C_m \frac{dV}{dt} = -g_m(V - V_{\text{rest}}) + I_{\text{inj}} \]

- Unfortunately, this model has been assumed to be a foundational part of theory

- Another unfortunate assumption is that all the complexity of the brain resides in the connectivity and synaptic weights

On the role of theory and modeling in neuroscience
D Levenstein, VA Alvarez, A Amarasingham, H Azab… - Journal of Neuroscience, 2023

Development of theoretical frameworks in neuroscience: a pressing need in a sea of data
The ideal capacitor does not exist

- The ideal capacitor is not physically possible
- Any basic electromagnetism book explains the assumptions to come up with the ideal capacitor

History dependence & power law charging

The charging of a capacitor follows a power-law!!!!!!!

Constant phase element

ELECTRIC PHASE ANGLE OF CELL MEMBRANES

By KENNETH S. COLE
(From the Department of Physiology, College of Physicians and Surgeons, Columbia University, New York)

(Accepted for publication, April 4, 1932)

Power law & history dependence in dielectrics

review article

The ‘universal’ dielectric response
A. K. Jonscher*

Curie J.
Recherches sur le pouvoir inducteur spécifique et sur la conductibilité des corps cristallisés

The mathematical object that gives constant phase and power-law behavior

• The ideal capacitor is

\[ I = C \frac{dV}{dt} \]

• Which results in an exponential

• What is the mathematical structure that when solved provides a power-law and constant phase?

\[ I = C \frac{d^\eta V}{dt^\eta} \]

• Take a minute to stare at the equation and realize that it has values in this range: \( 0 < \eta < 1 \)
Fractional differentiation and history dependence

• The fractional derivative is

\[
\frac{d^nV}{dt^n} = \frac{1}{\Gamma(1 - \eta)} \int_0^t \frac{V'(\tau)}{(t - \tau)^\eta} d\tau
\]

• This is a non-local operator that has memory

• Some people call this the intrinsic memory trace

• Thus, the systems depends on time, not just on its previous state (break down of the Markovian assumption)
Fractional differentiation, constant phase, and input whitening

- The fractional derivative of a sine wave is
  \[
  \frac{d^n A \sin(2\pi \omega t)}{dt^n} = (2\pi \omega)^n A \sin(2\pi \omega t + \frac{\eta \pi}{2})
  \]

- If input follows a power-law, \( A = (2\pi \omega)^{-\beta} \)
  \[ I_n = (2\pi \omega)^{-\beta} \sin(2\pi \omega t) \]

- Then
  \[
  f = \frac{d^n I_n}{dt^n} = (2\pi \omega)^{-\beta}(2\pi \omega)^{n} \sin(2\pi \omega t + \frac{\eta \pi}{2})
  \]

- If \( \beta = \eta \)
  \[
  f = 1 \ast \sin(2\pi \omega t + \frac{\eta \pi}{2})
  \]

- Optimal coding of pink noise spectra

- **Most natural signals follow pink noise**
  - Texture
  - Sounds – natural, human-made
  - Odorants – natural plumes, food recipes
  - Images – color, intensity
The sensory World has power-law statistics

- Alternative argument: What is the mathematical transformation that optimally...
Power-law statistics in human signals

Levitin, Chordia, Menon, PNAS 2011

Naming things

Russel et al, In J Comput Vis 2008

Food recipes

Kinouchi, Diez-Garcia et al. (2008) New Journal
Computational advantages of fractional neurodynamics
Another way to generate fractional order dynamics

- Alternative argument: Fractional order dynamics arises from processes outside therm
- Diffusion equation turns into Anomalous diffusion
- The reaction diffusion equation is the foundation to model neuronal activity

\[
C_m \frac{dV}{dt} = -g_m(V - V_{rest})
\]

\[
C_m \frac{dV}{dt} = -g_m(V - V_{rest}) - \sum_{i=1}^{N} g_i(V - V_i)
\]

\[
g_i = m^a n^b z^c
\]

\[
\frac{dm_i}{dt} = \alpha(1 - m_i) - \beta m_i
\]
Spiking acceleration in L5 pyramidal cells

**Experiments**


**Fractional model**

History dependence in L2/3 pyramidal cells

Experiments

Fractional model

And several other results
From membrane voltage to spiking

- The fractional order is in voltage.
- Does spike time or firing rate have fractional order properties?
  - Coding strategies of neurons.

\[ C_m \frac{d^n V}{d t^n} = -g_m (V - V_{rest}) + I_{inj} \]

Is the fractional order reflected in the firing rate properties?

\[ f \sim \frac{d^n I_n}{d t^n} \]
Evidence of fractional differentiation of firing rate

- Fractional order differentiation has been shown in
  - Cortical
  - Brainstem
  - Weakly electric fish.
  - Insects

Our preliminary data on Purkinje cells show these cells perform a fractional order differentiation.
Fractional differentiation and signal whitening by the fLIF

Fractional differentiation

\[(2\pi\omega)^\eta \sin\left(2\pi\omega t + \frac{\eta\pi}{2}\right)\]

Noise whitening

\[\frac{1}{\omega^{0.4}}\]
Instantiating the circuits
A super capacitor is a fractional order differentiator.

Vazquez-Guerrero, Tuladhar et al. (2024) Scientific Reports
Importance for AI

• Electric elements with memory are known as memelements
  • Memristors, memcapacitors, meminductors
  • Memristor—The missing circuit element, L. Chua, IEE Trans on circuit theory, 1971

• Intense interest
  • Implement neuronal functions intrinsically in hardware
  • Lower energy consumption

• Most people care about memristors
  • Materials
  • Used to model synapses

• Capacitors can me many times more energetically efficient than resistors
Real capacitors as memcapacitors

- A memcapacitor is
  \[ q = C(x, v, t)v \]  
  and \[ D^1 x = f(x, v, t) \]
- Where \( q \) is charge, \( C \) is capacitance, and \( x \) is an internal variable such as the flux:
  \[ D^1 \phi = v \]

Assuming that \( c(\phi, v, t) = c_0 \phi \) then

\[
q = C_0 \int_0^t v d\tau \cdot v
\]

Check out any Chua publication for details
This is a hysteresis process pinched at the center
Fractional order capacitors and memcapacitors

- A fractional order differentiator and a memcapacitor

Vazquez-Guerrero, Tuladhar et al. (2024) Scientific Reports
A fractional leaky integrate-and-fire circuit

\[ C_m \frac{d^{\eta}V}{dt^{\eta}} = -g_m(V - V_{rest}) + I_{inj} \]

Super-capacitor

Based on the classic design by Carver Mead
The fLIF circuit vs model vs data

Fractional circuit

Fractional model

Experiments

Vazquez-Guerrero, Tuladhar et al. (2024) Scientific Reports
The fLIF circuit vs model vs data

**Fractional circuit**

Vazquez-Guerrero, Tuladhar et al. (2024) Scientific Reports

**Fractional model**


**Experiments**

Lundström et al., Nat. Neurosci. 2008
Fractional differentiation and input whitening

\[ f = \frac{d^n I_n}{dt^n} \rightarrow f = A(2\pi \omega)^{\eta} \sin \left(2\pi \omega t + \frac{\eta \pi}{2}\right) \]

Vazquez-Guerrero, Tuladhar et al. (2024) Scientific Reports
A fractional order Hodgkin-Huxley circuit

Super-capacitor stack 1 mF

Teka, Stockton et al. (2016) PLoS computational biology

Vazquez-Guerrero, Tuladhar et al. (2024) Scientific Reports
Compare the circuit to experiments

- The pyramidal cells of the ELL perform a fractional derivative of the sensory input.

- Fractional dynamics of these neurons controlled by a potassium conductance

- Recording multiple neuron using neuropixel probes.

- Collaboration with Maurice Chacron @ McGill

The HH model and real data from weakly electric fish

Fractional differentiation

Input whitening

Criticality

Vazquez-Guerrero, Tuladhar et al. (2024) Scientific Reports
Predicting neuronal responses in live fish

Vazquez-Guerrero, Tuladhar et al. (2024) Scientific Reports
• Complexity in time at single cell scale provides optimal coding usually attributed to networks
  • Neurons adapt!
  • There is more to computation than synaptic plasticity
  • Intrinsic excitability should be incorporated into neuromorphic studies and designs

• Fractional order memcapacitor networks could provide
  • Optimal encoding – natural and human made signals
  • Optimal energy – the capacitor is the most efficient electric element

• Fractional differentiation in neurons exists at multiple scales
  • It can be measured
  • It is not a metaphor exported to AI
Collaborators

NIH R01EB026939 (PI)
Unified theory of adaptation
• Maurice Chacron, McGill U
• Ahmed Elwakil, Sharja UAE
• Costas Psychalinos, Patras Greece

NSF EFRI-BRAID 2318139 (PI)
Fractional-order neuronal dynamics
• Andy Sarles, U Tennessee
• Christof Teuscher, Portland State
• Yuriy Pershin, U South Carolina

NSF PARTNER 2332744 (co-PI)
Neuro-Inspired AI for the Edge
• Duke-UTSA
End